Dualizing the distributivity number $\mathfrak{h}$?

Wolfgang Wohofsky

joint work in progress with Yurii Khomskii and Marlene Koelbing

Universität Wien (Kurt Gödel Research Center)

wolfgang.wohofsky@gmx.at

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A family $A \subseteq [\omega]^\omega$ is mad (maximal almost disjoint) if $|a \cap b| < \aleph_0$ for each distinct $a, b \in A$, and it is maximal with this property, i.e., for each $z \in [\omega]^\omega$ there is $a \in A$ with $|a \cap z| = \aleph_0$.

For two mad families A and B, we say that B refines A if for each $b \in B$ there is an $a \in A$ with $b \subseteq^* a$.

Definition (Distributivity number (of $\mathcal{P}(\omega)/\mathrm{fin}$))

$\mathfrak{h}$ is the least size of a collection of mad families such that there is no single mad family refining all of them.

Claudio Agostini looked at a poster of mine about refining systems of mad families at the YSTW 2023 in Münster and asked me the following question:

Question

What is the least size of a collection of mad families such that each mad family is refined by one member of the collection?

A family $A \subseteq [\omega]^\omega$ is mad (maximal almost disjoint) if $|a \cap b| < \aleph_0$ for each distinct $a, b \in A$, and it is maximal with this property, i.e., for each $z \in [\omega]^\omega$ there is $a \in A$ with $|a \cap z| = \aleph_0$.

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What is the least size of a collection of mad families such that each mad family is refined by one member of the collection?

In some sense, this question is asking for a number which is dual to $\mathfrak{h}$.

Let us consider things in a more general setting:

A relational system is a triple $(\mathcal{X}, \mathcal{Y}, \sqsubseteq)$, where $\mathcal{X}$ and $\mathcal{Y}$ are sets, and $\sqsubseteq \subseteq \mathcal{X} \times \mathcal{Y}$ is a relation.

Recall the corresponding bounding number and dominating number:

$\mathfrak{b}(\mathcal{X}, \mathcal{Y}, \sqsubseteq) := \min\{|X| : X \subseteq \mathcal{X} \text{ unbounded}\}$
($X \subseteq \mathcal{X}$ is *unbounded* $:\iff$ there is no $y \in \mathcal{Y}$ with $x \sqsubseteq y$ for all $x \in X$)

$\mathfrak{d}(\mathcal{X}, \mathcal{Y}, \sqsubseteq) := \min\{|Y| : Y \subseteq \mathcal{Y} \text{ dominating}\}$
($Y \subseteq \mathcal{Y}$ is *dominating* $:\iff$ for each $x \in \mathcal{X}$ there is $y \in \mathcal{Y}$ with $x \sqsubseteq y$)

Well-known example:
$\mathfrak{b} = \mathfrak{b}(\omega^\omega, \omega^\omega, \leq^*)$
$\mathfrak{d} = \mathfrak{d}(\omega^\omega, \omega^\omega, \leq^*)$

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Let us now rephrase $\mathfrak{h}$ and its dual version:

$$\mathfrak{h} = \mathfrak{b}(\text{madfam}, \text{madfam}, \leftarrow_{\mathit{ref}})$$

Question (Claudio Agostini)

What is $\mathfrak{d}\mathfrak{h} := \mathfrak{d}(\text{madfam}, \text{madfam}, \leftarrow_{\mathit{ref}})$?

Thanks to discussions with Aleksander Cieślak a bit more than two weeks ago in Vienna, I realized that this is not the only way to "dualize" $\mathfrak{h}$...

...it depends on how we "define" $\mathfrak{h}$:

Fact

$\mathfrak{h} = \mathfrak{h}^\flat := \mathfrak{b}(\text{madfam}, [\omega]^\omega, \leftarrow_{\mathit{sel}})$

Let us dualize this version:

Definition

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Why $\flat$?

Recall:

$\mathfrak{dh} := \mathfrak{d}(\mathrm{madfam}, \mathrm{madfam}, \leftarrow_{\mathit{ref}})$

$\mathfrak{dh}^\flat := \mathfrak{d}(\mathrm{madfam}, [\omega]^\omega, \leftarrow_{\mathit{sel}})$

Note that, trivially,

$\mathfrak{dh} \leq 2^\mathfrak{c}$, but

$\mathfrak{dh}^\flat \leq \mathfrak{c}$

…i.e., in some sense, $\mathfrak{dh}^\flat$ is the lowered/flat/minor version of $\mathfrak{dh}$…

Let us generalize the definitions to arbitrary forcings:

$\mathfrak{h}(\mathbb{P}) = \mathfrak{b}(\mathrm{macs}(\mathbb{P}), \mathrm{macs}(\mathbb{P}), \leftarrow_{ref})$

$\mathfrak{d}\mathfrak{h}(\mathbb{P}) = \mathfrak{d}(\mathrm{macs}(\mathbb{P}), \mathrm{macs}(\mathbb{P}), \leftarrow_{ref})$

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Fact

$\mathfrak{h} = \mathfrak{h}(\mathcal{P}(\omega)/fin) = \mathfrak{h}^\flat(\mathcal{P}(\omega)/fin)$ *(uses homogeneity of $\mathcal{P}(\omega)/fin$)*

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Let us generalize the definitions to arbitrary forcings:

$\mathfrak{h}(\mathbb{P}) = \mathfrak{b}(\mathrm{macs}(\mathbb{P}), \mathrm{macs}(\mathbb{P}), \leftarrow_{ref})$

$\mathfrak{dh}(\mathbb{P}) = \mathfrak{d}(\mathrm{macs}(\mathbb{P}), \mathrm{macs}(\mathbb{P}), \leftarrow_{ref})$

$\mathfrak{h}^\flat(\mathbb{P}) = \mathfrak{b}(\mathrm{macs}(\mathbb{P}), \mathbb{P}, \leftarrow_{sel})$

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$\mathfrak{dh} = \mathfrak{dh}(\mathcal{P}(\omega)/fin)$

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Fact

- $\mathfrak{dh}(\mathbb{P}) \leq 2^{\mathbb{P}}$ *(for any forcing $\mathbb{P}$)*
- *if $\mathbb{P}$ has the λ^+-c.c., then $\mathfrak{dh}(\mathbb{P}) \leq |\mathbb{P}|^{\lambda}$*
- *for c.c.c. forcings on the reals (such as Cohen, random, Hechler), we have $\mathfrak{dh}(\mathbb{P}) \leq \mathfrak{c}^{\omega} = \mathfrak{c}$*

Lemma

If $\mathbb{P}$ has an antichain of size κ, then $\kappa < \mathfrak{dh}(\mathbb{P})$.

Corollary

- For non-c.c.c. forcings on the reals such as Sacks, Miller, Laver, Mathias, Silver, Full-miller, and also $\mathcal{P}(\omega)/\text{fin}$ (more on this later), we have $\mathfrak{c} < \mathfrak{dh}(\mathbb{P}) \leq 2^{\mathfrak{c}}$.
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- For c.c.c. forcings on the reals such as Cohen, random, Hechler, etc., we have $\omega < \mathfrak{dh}(\mathbb{P}) \leq \mathfrak{c}$.

Instead of maximal antichains of $\mathbb{P}$ also open dense sets of $\mathbb{P}$ can be used.

Let $\mathrm{opd}(\mathbb{P})$ denote the filter generated by the sets open dense in $\mathbb{P}$; equivalently, $D \subseteq \mathbb{P}$ is in $\mathrm{opd}(\mathbb{P})$ if and only if for each $p \in \mathbb{P}$ there is $q \leq p$ such that $r \in D$ for all $r \leq q$.

Fact

$$\mathfrak{h}(\mathbb{P}) = \mathfrak{b}(\mathrm{opd}(\mathbb{P}), \mathrm{opd}(\mathbb{P}), \supseteq) = \mathrm{add}(\mathrm{opd}(\mathbb{P}))$$

$$\mathfrak{h}^\flat(\mathbb{P}) = \mathfrak{b}(\mathrm{opd}(\mathbb{P}), \mathbb{P}, \ni) = \mathrm{cov}(\mathrm{opd}(\mathbb{P}))$$

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Let $\mathbb{P}$ be a tree forcings on 2^ω (or ω^ω or $[\omega]^\omega$). For a tree $T \in \mathbb{P}$, let

$$[p] := \{x \in 2^\omega : x{\restriction}n \in p \text{ for each } n \in \omega\}$$

the body of p (i.e., the set of branches through p).

Let p^0 denote the Marczewksi-null ideal associated to $\mathbb{P}$:

Definition

$p^0 := \{X \subseteq 2^\omega : \forall p \in \mathbb{P}\ \exists q \leq p \text{ such that } [q] \cap X = \emptyset\}$

Lemma

$\mathrm{cof}(p^0) \leq \mathfrak{dh}(\mathbb{P})$

Lemma

$\mathrm{non}(p^0) \leq \mathfrak{dh}^\flat(\mathbb{P})$

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$\mathcal{P}(\omega)/\text{fin}$

$\mathcal{P}(\omega)/\text{fin}$ is not an actual tree forcing, but let us treat the conditions as if it were, define "bodies" of conditions, and define a "Marczewski-style ideal":

For $a \in [\omega]^\omega$, let $\langle a \rangle := \{c \in [\omega]^\omega : c \subseteq^* a\}$.

Definition

$p\omega^0 = \{X \subseteq [\omega]^\omega : \forall \langle a \rangle\ \exists \langle b \rangle \subseteq \langle a \rangle\ (\langle b \rangle \cap X = \emptyset)\}$

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Lemma

$\mathfrak{c} < \operatorname{cof}(p\omega^0) \leq \mathfrak{dh}$

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Lemma

$\mathfrak{c} < \operatorname{cof}(p\omega^0) \leq \mathfrak{d}\mathfrak{h}$

r^0 (the Marczewski-null ideal for Mathias forcing)

...also called "Ramsey null" ideal or "nowhere Ramsey" ideal...

Lemma (Plewik? (where $\mathrm{add}(r^0) = \mathrm{cov}(r^0) = \mathfrak{h}$ proved))

$p\omega^0 = r^0$

Corollary

$\mathrm{cof}(r^0) \leq \mathfrak{dh}$

Also:

$\mathfrak{c} < \mathrm{cof}(r^0) \leq \mathfrak{dh}(\mathit{Mathias})$

$\mathfrak{c} < \mathrm{cof}(s^0) \leq \mathfrak{dh}(\mathit{Sacks})$

$\mathfrak{c} < \mathrm{cof}(\ell^0) \leq \mathfrak{dh}(\mathit{Laver})$

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Recall:

Lemma

$\mathrm{non}(p^0) \leq \mathfrak{dh}^\flat(\mathbb{P})$

Therefore, we get the following:

Lemma

$\mathrm{non}(p\omega^0) \leq \mathfrak{dh}^\flat \leq \mathfrak{c}$

But, as usual for non-c.c.c. "tree" forcings (in fact, due to $\mathfrak{c}$-sized antichains with disjoint bodies), we have:

Fact

$\mathrm{non}(p\omega^0) = \mathfrak{c}$

Corollary (the variant I dicussed with Alek)

$\mathfrak{dh}^\flat = \mathfrak{c}$

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For those who are interested in fresh functions and/or can remember past talks of mine about fresh function spectra etc.:

Lemma

$\mathrm{FRESH}(\mathbb{P}) \subseteq [\mathfrak{h}(\mathbb{P}), \mathfrak{d}\mathfrak{h}^\flat(\mathbb{P})]_{Reg}$.

Recall from some other talk (uses the base matrix theorem):

$\mathrm{FRESH}(\mathcal{P}(\omega)/\mathrm{fin}) = [\mathfrak{h}(\mathbb{P}), \mathfrak{c}]_{Reg}$.

Corollary (again, unnecessarily complicated)

$\mathfrak{d}\mathfrak{h}^\flat = \mathfrak{c}$

Cohen forcing $\mathbb{C}$

Let c^0 denote the ideal of nowhere dense subsets of 2^ω.

Lemma (from general lemma above)

$\mathrm{cof}(c^0) \leq \mathfrak{dh}(\mathbb{C}) \leq \mathfrak{c}$

In fact: $\mathrm{cof}(c^0) = \mathfrak{dh}(\mathbb{C})$!!!???

Theorem (Fremlin?; Balcar-Hernández-Hernández-Hrušák?)

$\mathfrak{dh}(\mathbb{C}) = \mathrm{cof}(\mathcal{M})$

Hechler forcing: $\mathfrak{dh}(\mathbb{D}) = \mathfrak{c}$

Eventually different forcing: $\mathfrak{dh}(\mathbb{E}) = \mathfrak{c}$

. . . same for filter-Laver for analytic filter. . .

Random: $\mathrm{cof}(\mathcal{N}) \leq \mathfrak{dh}(\mathbb{B}) \leq \mathfrak{c}$ . . . so what is $\mathfrak{dh}(\mathbb{B})$?

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Recall:

$\mathrm{cof}(p^0) \leq \mathfrak{dh}(\mathbb{P}) \leq |\{A \subseteq \mathbb{P} : A \text{ is a maximal antichain}\}|$

Question

Is it consistent that $\mathrm{cof}(p^0) < \mathfrak{dh}(\mathbb{P})$ for some $\mathbb{P}$?

Thank you for your attention and enjoy the Winter School...

Vienna, Augarten, 3rd December 2020

Thank you for your attention and enjoy the Winter School...

Vienna, Old KGRC (Josephinum), 9th April 2020